6. Problem sheet for Set Theory, Winter 2012

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Problem 19. (Fixed points)

(a) Show that $\mathrm{Ord}(x)$ if and only if $\mathrm{Trans}(x)$ and x is linearly ordered by $\in$.

(b) Let $F\colon Ord \to Ord$, $F(\alpha) = 2^\alpha$ (ordinal exponentiation). Show that $2^\omega = \omega$, i.e. ω is a fixed point of F.

(c) Show that every strictly increasing continuous function $F\colon Ord \to Ord$ has arbitrarily large fixed points, i.e. if $\alpha < \beta \to F(\alpha) < F(\beta)$ for all $\alpha, \beta \in Ord$ and $F(\lambda) = \sup_{\alpha<\lambda} F(\alpha)$ for all limits $\lambda \in Ord$, then $\forall\gamma\exists\delta > \gamma$ $F(\delta) = \delta$.

Problem 20. (Well-orders) A linearly ordered set $(x, <)$ is *well-ordered* if every non-empty $y \subseteq x$ has a $<$-minimal element. Show:

(a) If $(x, <)$ is a well-ordered set, then there is a unique ordinal α such that $(\alpha, \in)$ and $(x, <)$ are isomorphic. Use the recursion $F(\beta) = \min(x \setminus range(F \upharpoonright \beta))$ for $x \setminus range(F \upharpoonright \beta) \neq \emptyset$.

(b) If $(x, <_x)$ and $(y, <_y)$ are both well-ordered, then the lexicographical product $(x \times y, <_{lex})$ is well-ordered. We define $(a, b) <_{lex} (a', b') :\leftrightarrow (a <_x a') \vee (a = a' \wedge b <_y b')$.

(c) $\alpha \cdot \beta$ is the order type of the lexicographical product of $(\beta, \in)$ and $(\alpha, \in)$ for $\alpha, \beta \in Ord$.

Problem 21. (Hausdorff Maximality Principle) Show that in the theory ZF the axiom of choice is equivalent to the Hausdorff Maximality Principle which says: for every partial order $(P, \leq) \in V$ there is an inclusion maximal chain X in $(P, \leq)$, i.e. X is a chain and if $Y \supseteq X$ is a chain in $(P, \leq)$ then $Y = X$.

Problem 22. (Embedding into $\mathbb{Q}$) Suppose γ is a countable ordinal. Show that there is an order-preserving injection $f\colon \gamma \to \mathbb{Q}$, i.e. $\forall\alpha < \beta < \gamma \;\; f(\alpha) < f(\beta)$.

There are 6 points for each problem. Please hand in your solutions on Monday, November 19 before the lecture.